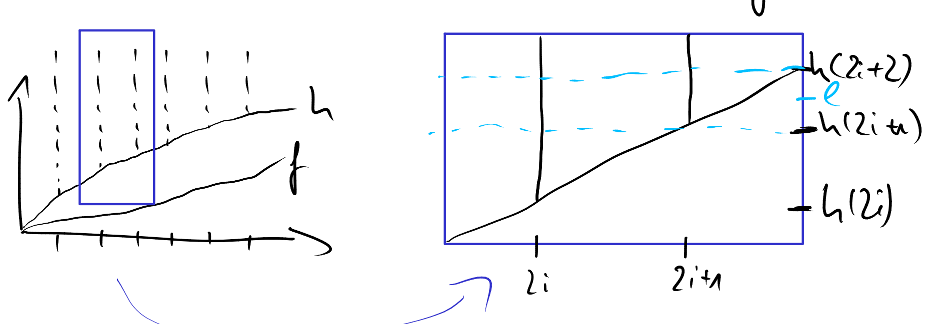
(Main) Lemma Let $G=(g_i)_{i\in\omega}$ be $\mathbb{P}(\{\omega\}\otimes u)$ ($G(\{\omega\})\otimes u$ or $\mathcal{S}(\{\omega\}\otimes u)$) -generic real.
Let $\mathbb{Q}\in V[G]$ be an ω^ω-bounding forcing, H $\mathbb{Q}$-generic.
Then $V[G][H]\models u$ does not extend to a P-point.

Proof:

- To start, we work in $V[G][H]$.
- ~~Assume~~ $\tilde{u}\geq u$ is a P-point.
- Then $\tilde{u}$ can decide each g_i $(i\in\omega)$
- ! Assume for simplicity that $g_i\in\tilde{u}$ $\forall i\in\omega$ **!** — This is a real simplification! I'll discuss the general case at the end
- Then let $A_i := g_{2i}\cap g_{2i+1}\in\tilde{u}$ $(i\in\omega)$ — in the general case, we need more complicated intervals than $[2i,2i+1]$
- Since $\tilde{u}$ is a P-point, we find a pseudointersection $A\in\tilde{u}$ with $\forall i$ $A\subseteq^* A_i$
- In other words, there's $f:\omega\to\omega$ with $\forall i$ $A\setminus f(i)\subseteq A_i$
  i.e. $\bigcap_{i\in\omega}(A_i\cup f(i))\in\tilde{u}$
- Since $\mathbb{Q}$ & $\mathbb{P}(\{\omega\}\otimes u)$ [remember the "detour"] are ω^ω-bounding; wlog $f\in\omega^\omega\cap V$.
- So we have a statement in $V[G]$: "$\bigcap_{i\in\omega}(A_i\cup f(i))$ is compatible with u" !
- Take $g\in\mathbb{P}(\{\omega\}\otimes u)$ which forces this.

We will build a stronger condition that forces $\bigcap_{i\in\omega} A_i\cup f(i)$ to be finite!

- As noted in the "detour", we can assume $\mathrm{split}(g)=\prod_{i\in\omega}\{i\}\times(B\setminus h(i))$ for some $B\in u$, $h\in\omega^\omega\cap V$
- Of course, we may also assume $f\leq h$.
- But now we're ready to define $\tilde{g}=(\tilde{g}_i)_{i\in\omega}\leq g=(g_i)_{i\in\omega}$.
  To motivate our choices, let's draw a picture of the split g_i, the values not yet decided.



To get "$\ell\in A_i$" both g_{2i} and g_{2i+1} must agree, i.e. $g_{2i}(\ell)=g_{2i+1}(\ell)=1$