Now we're ready for some fusion!

Recall: $\mathbb{P}$ is ω^ω-bounding if $\forall G$ generic $\forall f \in V[G] \cap \omega^\omega$ $\exists g \in \omega^\omega \cap V$: $f \leq g$.

<u>Lemma</u> If u is a non-meagre P-filter, then $\mathbb{P}$ is ω^ω-bounding.

<u>Proof</u>: ∘ Let $p_0 \Vdash \dot{f} \in \omega^\omega$.

∘ We describe a strategy for I in the P-point game.

I	split p_0	...	$\mathrm{split}(p_{(a_0,\dots,a_{n-1})})$		...
II		$a_0 \subseteq \mathrm{split}\, p_0$	...	$a_n \subseteq \mathrm{split}(p_{(a_0,\dots,a_{n-1})})$	...

doing the following

1) $p_{(a_0,\dots,a_{i+1})} \leq p_{(a_0,\dots,a_i)}$ $\forall i < n-1$

2) $\bigcup_{j<i} a_j \subseteq \mathrm{split}\,(p_{(a_0,\dots,a_i)})$ $\forall i < n$

(We also keep track of) → 3) $H_{(a_0,\dots,a_i)} \subseteq_f \omega$ such that $p_{(a_0,\dots,a_i)} \Vdash \dot{f}(i) \in H_{(a_0,\dots,a_i)}$

[This is easy with the preceding "crucial fact" lemma]

∘ Now II can beat this strategy (since u is a non-meagre P-filter), i.e. there are $(a_i)_{i\in\omega}, (p_{(a_0,\dots,a_i)})_{i\in\omega}, (H_{(a_0,\dots,a_i)})_{i\in\omega}$ such that

1) $p_{(a_0,\dots,a_i)} \Vdash \dot{f}(i) \in H_{(a_0,\dots,a_i)}$

2) $\mathrm{split}(p_{(a_0,\dots,a_i)}) \supseteq \bigcup_{j<i} a_j$

3) $\bigcup_{i\in\omega} a_i \in u$

∘ But then $p := \bigcap_{i\in\omega} p_i \in \mathbb{P}$ $\left[\mathrm{split}\, p \supseteq \bigcup_{n\in\omega} a_n \in u\right]$

& $p \Vdash \dot{f}(i) \in H_{(a_0,\dots,a_i)}$ for all i $\left[p \leq p_{(a_0,\dots,a_i)}\right]$

∘ In particular, $p \Vdash \dot{f}(i) \leq \max H_{(a_0,\dots,a_i)}$

so $g: \omega \to \omega$, $n \mapsto \max H_{(a_0,\dots,a_n)}$ is as desired.

□

The proof for properness is similar, but we need the definition first...