Imagine we played as follows

| | | | | | |
|---|---|---|---|---|---|
| I | $\text{split}(p_0)$ | | $\text{split}(p_1)$ | | ... |
| II | | $a_0 \subseteq \text{split}(p_0)$ | | $a_1 \subseteq \text{split}(p_1)$ | ... |

with $p_n \Vdash \dot{j}(n) = k_n$

Then if $u$ is a non-meagre P-filter, II could beat the strategy! In that case $\bigcup_{i\in\omega} a_i \in u$ — if only we could guarantee the splitting levels survive, i.e. $\bigcup_{i<n} a_i \subseteq \text{split}\, p_n$, then new $p_n$ would be a condition!

Of course we cannot hope for this — just imagine $j$ was the generic real!

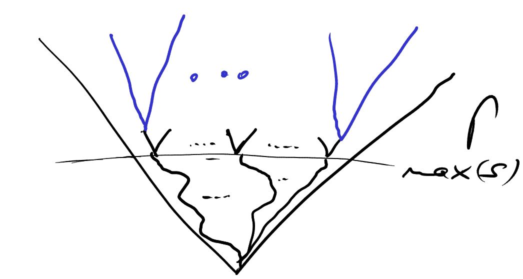
But we need far less anyway...

__Lemma__ $p \in \mathbb{P}$, $p \Vdash \dot{x} \in V$, $s \subseteq_{fin} \text{split}\, p$

$\Rightarrow \exists H \subseteq_{fin} V, q \leq p : q \Vdash \dot{x} \in H$

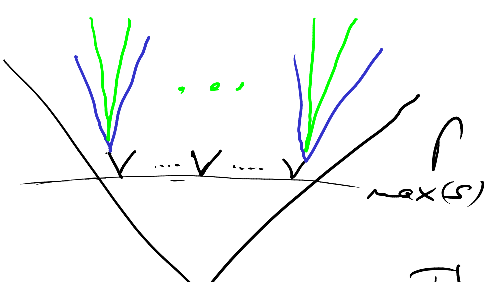
$\& \; s \subseteq_{fin} \text{split}\, q$

__Proof__: Let's prove this by picture (this is simply amalgamation)




Take each subtree} $p_i \leq p$ [these are finitely many, say $j$-many]

For Grigorieff: Do lexicographic iteration to get finer and finer V.




Refine each $p_i$ } to $q_i \leq p_i$ }

to have $q_i \Vdash \dot{x} = x_i$ for some $x_i \in V$.

Then $q = \bigcup_{i<j} q_i \leq p$ and $q \Vdash \dot{x} \in \{x_i \mid i<j\}$ # □