In an earlier post I gave a short introduction to an interesting finite semigroup. This semigroup could be found in the 2×2 matrices over $\mathbb{Q}$.

When I met with said friend, one natural question came up: what other semigroups can we find this way?

The first few simple observations we made were

Remark 0.0.1
- If either A or B is the identity matrix I_2 or the zero matrix 0_2 the resulting semigroup will contain two elements with an identity or a zero element respectively.
- In general, we can always add I_2 or 0_2 to the semigroup generated by A and B and obtain a possibly larger one.
- A, B generate a finite semigroup iff AB is of finite order (in the sense that the set of its powers is finite).
- AB has finite order iff its (nonvanishing) eigenvalue is ± 1.
- For A of rank 1 we may assume (by base change) that A is one of the two matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

So, as a first approach we thought about the following question.

Question 0.0.2 If we take A to be one of the above, what kind of options do we have for B, i.e., if B is idempotent and A, B to generate a finite semigroup.

Thinking about the problem a little and experimenting with Macaulay 2 we ended up with the following classification

Proposition 0.0.3 *For* $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *the solutions for* B *being of rank one consist of four one-dimensional families, namely (for* $x \in \mathbb{Q}$*)*

$$F_1(x) = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}, F_2(x) = \begin{pmatrix} 1 & 0 \\ x & 0 \end{pmatrix}, F_3(x) = \begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix}, F_4(x) = \begin{pmatrix} 0 & 0 \\ x & 1 \end{pmatrix}.$$

Additionally, we have four special solutions

$$G_1 = \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}, G_2 = \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}, G_3 = \begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}, G_4 = \begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix}.$$

We can also describe size and the algebraic structure.

- A with F_1 (F_2) generates a right (left) zero semigroup (hence of size 2, except for $x = 0$).
- A with F_3 or F_4 generates a semigroup with AB nil-potent (of size 4, except for $x = 0$, where we have the null semigroup of size 3).

- A with G_i generate (isomorphic) semigroups of size 8. These contain two disjoint right ideals, two disjoint left ideals generated by A and B respectively.

Luckily enough, we get something very similar from our alternative for A.

Proposition 0.0.4 *In case* $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ *the solutions for* B *being of rank one consist of five one-dimensional families namely (for* $x \in \mathbb{Q}$*)*

$$H_1(x) = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}, H_2(x) = \begin{pmatrix} x+1 & x \\ -x-1 & -x \end{pmatrix},$$
$$H_3(x) = \begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix}, H_4(x) = \begin{pmatrix} -x+1 & -x+1 \\ x & x \end{pmatrix},$$
$$H_5(x) = \begin{pmatrix} -x+1 & -x-1-\frac{2}{x-2} \\ x-2 & x \end{pmatrix}, \ x \neq 2.$$

As before we can describe size and structure.

- A with H_1 (H_2) generates a right (left) zero semigroup (as before).
- A with H_3 or H_4 generates a semigroup with AB nilpotent (as before).
- A with H_5 generates the same 8 element semigroup (as before).

Finally, it might be worthwhile to mention that the seemingly missing copies of the 8 element semigroup are also dealt with; e.g. $-G_i$ generates the same semigroup as G_i etc.

It is striking to see that the orders of all finite semigroups generated by rational idempotent two by two matrices are either $2^k, 2^k+1$ or 2^k+2

At first sight it seems strange that we cannot find other semigroups with two generators like this. As another friend commented, there's just not enough space in the plane. I would love to get some geometric idea of what's happening since my intuition is very poor. But that's all for today.